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Technical Notes

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Bayesian Framework for Calibration of Gas Turbine Simulator

P. M. Tagade,* K. Sudhakar,† and S. K. Sane†
Indian Institute of Technology, Bombay, Mumbai 400 076, India

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Nomenclature

A = area, m^2
CN = corrected speed
F = thrust, N
f = probability density function
I = polar moment of inertia, $kg \cdot m^2$
N = spool speed, rps
P = pressure, Pa
PR = pressure ratio
Pr = probability
R = gas constant
STR = scaled temperature ratio
SW = scaled mass flow rate
T = simulator
TR = temperature ratio
V = lumped volume, m^3
W = mass flow rate, kg/s
W_s, TR_s = scaling parameters
$\bar{x}$ = vector of control input
Y = system response
α, λ = hyperparameters of the Gaussian process
γ = isentropic index
ε = measurement uncertainty
ζ = true system response
θ = vector of parameters
μ = mean
Σ = covariance matrix
σ = standard deviation
τ = torque, $N \cdot m$

Subscripts

C = compressor
e = experiment
F = fuel
NZ = nozzle
S = scaling
T = turbine
t = true
U = upstream
3 = compressor exit
5 = turbine exit

Introduction

EXPERIMENTAL observations and simulation models are replete with uncertainties that limit understanding of the physical process [1]. Well-defined methods are adopted by the research community to express uncertainty associated with experimental observations [2], but the same is not true with modeling and simulation. The importance of uncertainty analysis in credibility assessment of computer simulation is elaborated by Mehta [3]. An account of uncertainty in every stage of modeling and simulation process is given by Oberkampf et al. [4]. Uncertainty is classified as aleatory uncertainty, which emanates from inherent variability of the system and epistemic uncertainty, which stems from the lack of knowledge about the system. Probability theory is used by researchers to represent aleatory uncertainty. However, frequentist probability theory cannot be used to represent epistemic uncertainty, and researchers have proposed the use of the Bayesian probability theory [5].

Prediction of induced uncertainty in system response due to uncertainties in various phases of the modeling and simulation process is known as uncertainty propagation. Calibration is a method of reducing uncertainties associated with the simulation model using experimental observations of the system response. A probabilistic framework that encompasses both the uncertainty propagation and the calibration and where probability is defined based on the *degree of belief* of an individual is termed as Bayesian framework. See Glimm and Sharp [6] for application of Bayesian framework in varied fields. Kennedy and O'Hagan [7] have proposed Gaussian-process-based Bayesian framework, where uncertainty in model structure and uncertainty due to a limited number of simulations is also considered. See Higdon et al. [8], Goldstein and Rougier [9], and Bayarri et al. [10] for further investigations of Bayesian framework. See Trucano et al. [11] for an extensive review of the research work in this field.

The present work probes a Bayesian framework to calibrate the simulator of a gas turbine engine. The simulator considered is based on low-fidelity models, with most of the components described using performance maps and other parameters. Maps and parameters are assumed to be unknown or poorly known initially, resulting in uncertain predictions. Methodology to identify, quantify, and reduce such uncertainties using calibration is demonstrated. The epistemic nature of these uncertainties require Bayesian probability theory for representation. Roth et al. [12] have discussed the intricacies of calibration methodology for a gas turbine engine simulator. Roth et al. [13] initially proposed the use of Bayesian inference for calibration but ultimately used the minimum variance optimum estimator method [14]. In the present paper, authors have demonstrated the use of Bayesian framework for the calibration of a gas turbine simulator in the presence of an uncertain compressor map. The choice of the compressor map for demonstration of the methodology is motivated mainly by critical role of the compressor map in performance prediction of the gas turbine engine. Methodology can similarly be applied to other component maps and parameters.

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*Research Scholar, Department of Aerospace Engineering.
†Professor, Department of Aerospace Engineering.

Authors propose a two-step probabilistic approach for inference of the compressor map. The first step is the Bayesian calibration of scaling parameters using a single experimental observation, and it results in a global update of the compressor map. In the second step, the compressor map is represented as a nonstationary Gaussian process with the mean given by an updated map of the first step. The proposed method is demonstrated using a hypothetical test bed engine, which is specified through a set of deterministic parameters and component maps. This hypothetical test bed engine can be "run" using the engine simulator with "sensors" that can tap response with typical sensor errors. Hypothetical test bed steady-state response is used for Bayesian calibration.

Bayesian Framework

The relationship between true physical system response $\zeta(\bar{x})$ and observed system response $Y_e(\bar{x})$ at deterministic input conditions $\bar{x}$ is given by

$$Y_e(\bar{x}) = \zeta(\bar{x}) + \varepsilon(\bar{x}) \tag{1}$$

where $\varepsilon(\bar{x})$ is uncertainty in experimental observations. The simulator $T(\bar{x}, \bar{\theta})$ predicts the response of the system on specification of the deterministic inputs $\bar{x}$ and uncertain parameters $\bar{\theta}$. The relationship between true system response and simulator is given by

$$\zeta(\bar{x}) \equiv T(\bar{x}, \bar{\theta}_t) \tag{2}$$

where $\bar{\theta}_t$ is the true value of the parameters $\bar{\theta}$, which can be inferred through Bayesian calibration. The advantages of Bayesian calibration methodology are discussed by Trucano et al. [11]. Bayesian calibration is a methodology to update prior distribution of the parameters based on comparison of simulator outputs with observations using Bayes' theorem [15].

Given observed system response $Y_e(\bar{x})$, Bayesian inference of the parameters $\bar{\theta}$ is given by

$$f_\theta(\bar{\theta}|Y_e(\bar{x})) \propto Pr(Y_e(\bar{x})|T(\bar{x}, \bar{\theta})) \times f_\theta(\bar{\theta}) \tag{3}$$

The likelihood $Pr(Y_e(\bar{x})|T(\bar{x}, \bar{\theta}))$ is specified based on experimental uncertainty $\varepsilon(\bar{x})$. Conjugate priors [15] are used to specify prior uncertainty in parameters $\bar{\theta}$. Markov chain Monte Carlo [16] (MCMC) method is used to sample from posterior distribution.

Gas Turbine Engine Model

The Bayesian framework is demonstrated for a single spool turbojet engine simulator. A simple lumped volume-based model is used to simulate the engine [17]. See Sanghi et al. [18] for review of different models used to simulate the gas turbine engine. The framework is general in nature and can be used for any model without change in the methodology. The model is a set of three ordinary differential equations that simulates response of the engine on specification of control inputs $\bar{x} = (W_F, A_{NZ})$ and component maps. A mathematical model for static element is based on aerothermodynamic relation and steady-state component maps. Rotor dynamics is modeled using conservation of angular momentum, which is given by

$$\dot{N} = \frac{\tau_T - \tau_C}{2 \times \pi \times I} \tag{4}$$

Gas dynamics is modeled using conservation of mass and energy equations [19]. The model is simplified by assuming that all the processes are isentropic and Mach number is low across all the lumped volumes. Gas dynamics is, thus, modeled using

$$\dot{P} = \frac{\gamma \times R \times T}{V}(W_U - W) \tag{5}$$

Compressor exit pressure P_3, turbine exit pressure P_5, and spool speed N are used as state variables.

Bayesian Calibration Method for Uncertain Compressor Map

A compressor map is a specification of corrected mass flow rate and temperature ratio as a function of pressure ratio and corrected speed, which is usually specified by the finite set of values with interpolation at other locations. Let $\overline{PR} = \{PR_i\}$ denote a vector of discrete pressure ratios with M elements and $\overline{CN} = \{CN_j\}$ be a vector of corrected speeds having N elements. Let $\bar{W} = \{W_{ij}\}$ denote a vector of corrected mass flow rate with $M \times N$ elements, where W_{ij} is corrected mass flow rate at each pair of $PR_i \in \overline{PR}$ and $CN_j \in \overline{CN}$. Thus, a mass flow rate map is represented by $\{\overline{CN}, \overline{PR}, \bar{W}\}$. The temperature ratio map is similarly represented by $\{\overline{CN}, \overline{PR}, \overline{TR}\}$.

Step One: Scaling Parameters

An arbitrary compressor map with high uncertainty is chosen to initiate the methodology representing a complete lack of knowledge about the map. It may be noted that current industry practices are such that designers do have a high degree of confidence on a compressor map even at this stage. In such a scenario, step one can be avoided, and methodology can be started from step two. Compressor map is scaled using design point corrected mass flow rate W_s and temperature ratio TR_s. Thus, the parameters are $\bar{\theta} = \{W_s, TR_s\}$. Scaled maps are obtained from elements of the compressor map using

$$SW_{ij} = \frac{W_{ij}}{W_s} \qquad STR_{ij} = \frac{TR_{ij} - 1}{TR_s - 1} \tag{6}$$

Probability distribution on the set of compressor maps is specified by the probability distribution on the scaling parameters.

Being a conjugate pair, likelihood and prior are specified using the Gaussian distribution. Thus, posterior is given by

$$\begin{aligned} &f_{W_s}(W_s|Y_e(W_F, A_{NZ})) \times f_{TR_s}(TR_s|Y_e(W_F, A_{NZ})) \\ &\quad \propto \exp\left(-\frac{\{Y_e(W_F, A_{NZ}) - T(W_F, A_{NZ}; W_s, TR_s)\}^2}{2 \times \sigma_Y^2}\right) \\ &\quad \times \exp\left(-\frac{(W_s - \mu_{W_s})^2}{2 \times \sigma_{W_s}^2}\right) \times \exp\left(-\frac{(TR_s - \mu_{TR_s})^2}{2 \times \sigma_{TR_s}^2}\right) \end{aligned} \tag{7}$$

Step Two: Gaussian Process Representation

Uncertainty in $\{W, TR\}$ at each $\{CN, PR\}$ can be specified by marginal Gaussian distribution of the joint distribution of a finite set of $\{W, TR\}$. This allows probabilistic representation of the compressor map as a Gaussian process [20]. Steady-state response of a gas turbine engine is a function of the local area of the compressor map. Thus, calibration results in localized update of the compressor map. This prompted the use of a nonstationary Gaussian process to represent the compressor map.

In the present work, the compressor map $\{\overline{CN}, \overline{PR}, \bar{\mu}_W, \bar{\mu}_{TR}\}$ obtained from the posterior mean scaling parameters of the first step is used as a prior mean map. Thus, $\{\mu_W, \mu_{TR}\}$ is the mean vector. On specification of the covariance matrix, probability of the compressor map is given by

$$\begin{aligned} Pr(\bar{W}, \overline{TR}) &\propto \exp\left(-\tfrac{1}{2} \times (\bar{W} - \bar{\mu}_W)^T \times \Sigma_W^{-1} \times (\bar{W} - \bar{\mu}_W)\right) \\ &\quad \times \exp\left(-\tfrac{1}{2} \times (\overline{TR} - \bar{\mu}_{TR})^T \times \Sigma_{TR}^{-1} \times (\overline{TR} - \bar{\mu}_{TR})\right) \end{aligned} \tag{8}$$

In the present work, the following covariance function is used:

$$\begin{aligned} &\mathrm{cov}(W_{CN_i, PR_i}, W_{CN_j, PR_j}) \\ &\quad = \sigma_i \sigma_j \exp(-(\lambda_{PR}(PR_i - PR_j)^2 + \lambda_{CN}(CN_i - CN_j)^2)) \end{aligned} \tag{9}$$

where σ denotes the standard deviation of marginal Gaussian distribution. The strength of correlation in each direction is controlled by the hyperparameter λ, which can be fixed using expert judgment or methods like maximum a posteriori estimation [21]. Let I_1 denote all the information contained in the Gaussian process.

Mass flow rate increases gradually with the decreasing pressure ratio and becomes constant at choking conditions. Denote this expert information by I_2, which can be used along with experimental observations for the Bayesian calibration. Using information I_2, experimental observations $Y_e(W_F, A_{NZ})$ and I_1 sequentially, posterior distribution of the compressor map is given by

$$Pr(\bar{W}, \overline{TR}|I_2, Y_e(W_F, A_{NZ}), I_1) \propto Pr(I_2|Y_e(W_F, A_{NZ}), \bar{W}, \overline{TR}, I_1) \times Pr(Y_e(W_F, A_{NZ})|\bar{W}, \overline{TR}, I_1) \times Pr(\bar{W}, \overline{TR}|I_1) \quad (10)$$

After the addition of an experimentally observed system response, the shape of the posterior mean map may not be acceptable to an expert due to its shape (see Fig. 1). Information I_2 contains the expert opinion that mass flow rate at points below a should be greater than or equal to that at a, with choking at the lowest pressure ratio. This information is provided to points below a by generating pseudo-system response, which is the mass flow rate at all pressure ratios below a, as shown in Fig. 1. Slope of the mass flow rate curve is a maximum near stall line and decreases gradually to zero at the choking condition. Assuming a linear change in slope, this information along with mass flow rate at a is used to generate the pseudo-system response below a. Uncertainty in the expert opinion is specified by posterior standard deviation of the mass flow rate at a. Any further information available from an expert can be incorporated in a similar fashion.

The posterior Gaussian process represents updated knowledge about the compressor map after addition of information through experimental observations and expert judgment. This updated knowledge is propagated to the system response using a Monte Carlo method. A large finite set of samples is collected from posterior distribution and system response is obtained at these samples. Confidence on the simulator is measured by Bayesian confidence bounds on predicted system response.

Results and Discussion

Methodology is demonstrated with five steady-state experimental observations of the hypothetical test bed engine, with epistemic uncertainty in the compressor map. Compressor exit total pressure (P_3), turbine exit total pressure (P_5), spool speed (N), and thrust (F) are used as observed system responses. Though the results are reported only for mass flow rate, similar results are obtained for the temperature ratio map also.

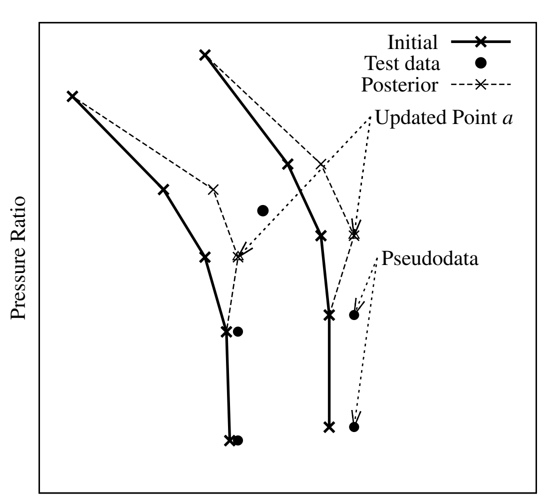


Fig. 1 Bayesian update using expert opinion.

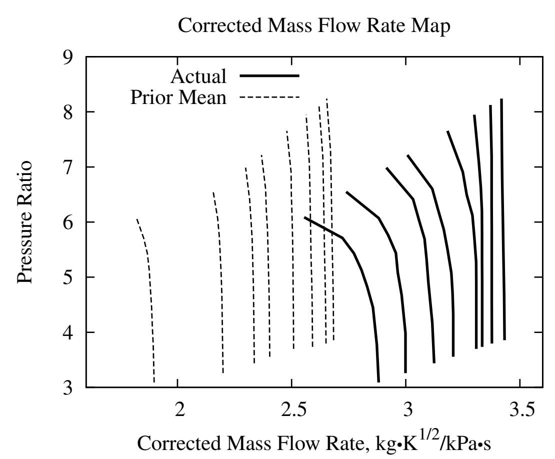
Table 1 Details of Bayesian analysis (step one)

Initial information		
Parameter	*Mean*	*Standard deviation*
Scaling mass flow, W_s	2.58	0.84
Scaling temperature ratio, TR_s	1.72	0.03
Experimental System Response		
Compressor exit pressure, P_3	671.7	6.7
Turbine exit pressure, P_5	177.5	1.7
Engine thrust, FN	8330.0	83.3
Spool Speed, N	100.0	1.0
After Bayesian update		
Scaling mass flow, W_s	3.35	0.1
Scaling temperature ratio, TR_s	1.85	0.006

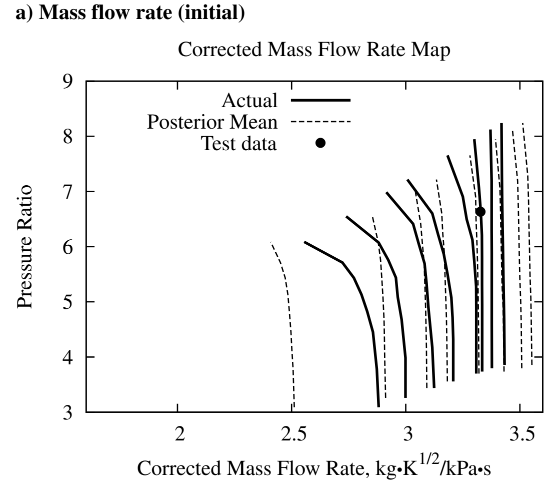
Step One: Scaling Parameter

Bayesian analysis is performed using MCMC algorithm with normal distribution as a proposal distribution. A total of 200,000 samples are collected after a burnout period of 100,000 samples. Uncertainty associated with system responses and details of the Bayesian analysis are given in Table 1.

In Fig. 2, the actual mass flow rate map is compared with a prior and globally updated posterior mean map after step one. The posterior mean map is close to the actual map, with a better match near the 100% speed line. Because scaling parameters are defined with respect to design point, it is recommended to use experimental observation near the design point.



a) Mass flow rate (initial)



b) Mass flow rate (updated)

Fig. 2 Comparison of mass flow rate map after Bayesian update (step one).

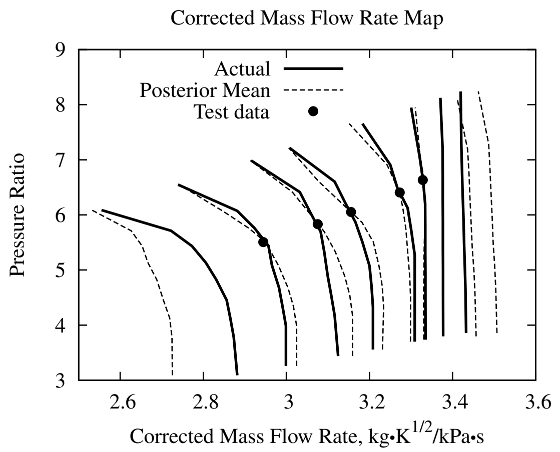


Fig. 3 Comparison of compressor map after Bayesian update (step two).

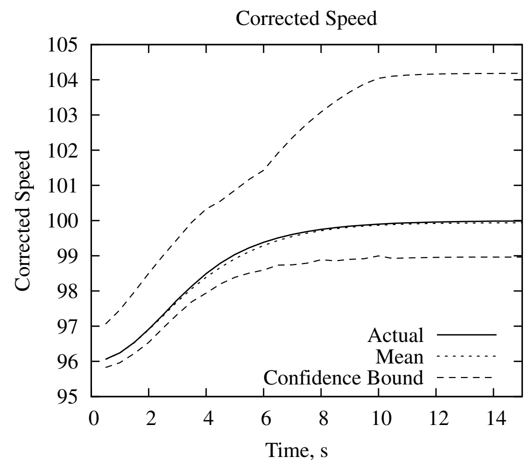


Fig. 5 Comparison of transient performance (corrected speed).

Step Two: Gaussian Process Representation

For the present investigation, $\lambda_{PR} = 1.0$ and $\lambda_{CN} = 1.2$ are used, which is a heuristic choice taking into account the computational difficulties involved in the inversion of a covariance matrix. Here, instead of a temperature ratio, the compressor exit temperature is used in the Bayesian analysis. In the present work, equal standard deviations of $\sigma_W = 0.5$ and $\sigma_{TR} = 2.0$ are used at all pressure ratios and corrected speeds, with 1% standard deviation in each system response.

Total 700,000 samples are collected from posterior distribution after an initial burnout period of 100,000 samples. Comparison of the actual compressor map with a posterior mean after Bayesian calibration is shown in Fig. 3. It is clearly evident from the figure that the proposed methodology of using experimental observation and expert judgment sequentially has resulted in a mean compressor map, which is close to an actual map. A close match is obtained at locations where experimental data are added. Points adjacent to these locations are updated because of the covariance in Gaussian process. Increased confidence on the compressor map after Bayesian analysis is evident from the contours of standard deviation shown in Fig. 4 for mass flow rate. The addition of information results in a decrease inbound on possible compressor maps and an increase in probability of the mean compressor map.

Uncertainty Propagation

Uncertainty in the posterior compressor map is propagated to the system response using a Monte Carlo method. A total of 200,000 samples are collected, and the transient is simulated for each sample. Transient analysis is performed with linear variation in the control input, and initial conditions are assumed to be completely known. 90% Bayesian confidence bound is predicted by neglecting 5% samples at tail. Comparison of transient performance along with 90% confidence bound for spool speed is shown in Fig. 5. Similar posterior confidence bounds can be obtained for any system response of interest.

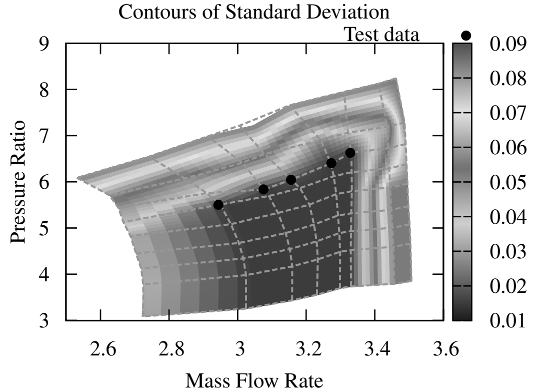


Fig. 4 Contours of standard deviation for mass flow rate.

Conclusions

Bayesian framework is proposed and demonstrated for the gas turbine modeling and simulation process in the presence of uncertainty. A two-step approach is proposed to calibrate the component maps using experimental observations and expert opinion. Framework is further used to predict confidence bounds on hitherto unobserved system responses. The first step of the methodology can be avoided if a component map with high confidence is provided in the beginning. Though a specific form of the covariance function is used in the present work, a different covariance function may be more suitable for other applications.

Proposed use of a nonstationary Gaussian process requires a large number of parameters. Because of the nonuniformity in information content of the compressor map, this situation could not be avoided. As present simulation is computationally inexpensive, high dimensionality of the Gaussian process parameters does not pose significant limitations. In the case of a computationally expensive simulator, statistical emulator-based methods [8] can be used without any change in the proposed methodology. The proposed framework is intended to aid decision makers while designing of gas turbine engine, especially in situations where prior expertise of engine design is not available.

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C. Tan
Associate Editor